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Transverse flow driven by walls oscillating along their normal

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When a pipe, connected to reservoirs at both ends, is subjected to an oscillating compression and expansion, fluid is alternately squeezed into and out of the reservoirs, causing a considerable amount of dissipation. This and two other related geometries (involving a symmetrical channel, and a pair of circular disks) are analysed for their flow patterns and energy dissipation, as a function of frequency. It is found that, under certain circumstances, the impedance due to transverse flow can greatly exceed the acoustical impedance (due to longitudinal flow).

1. Introduction

The calculations we shall describe came about as a result of an examination of the energy losses caused by viscous fluid flow in the active volume of a condenser microphone. Because energy loss adversely affects their efficiency, condenser microphones are designed to minimize this effect. Therefore, in the context of condenser microphones, our calculation can be employed to indicate when such losses might present a problem. However, the problem is so general, and the solution is so straightforward, that we believe our results to be of broader interest.

A condenser microphone or transmitter consists of a tightly stretched, electrically conducting, diaphragm which is placed a small distance h above a metallic backplate (Wente 1917). In the transmitter mode, a time-varying voltage difference between the diaphragm and backplate causes a time-varying force to act on the diaphragm, bringing it into motion, and thus generating sound in the fluid in which it is immersed. Indeed, this is the same principle employed in an electrostatic speaker. In the microphone mode, incident sound causes the diaphragm to move, thus changing the capacitance of the diaphragm-backplate system, and producing an electric current to or from a voltage source as the system attempts to maintain a fixed voltage difference between the diaphragm and the backplate. Viscous loss inevitably occurs when the fluid is forced about by the motion of the diaphragm. In at least one type of geometry, this viscous loss can be quite significant. It can occur if the backplate is not flat, but rather has holes or annular ridges cut in it, and the characteristic contact distance (the square root of the contact area) between diaphragm and backplate is much larger than their separation h. In this case, when the diaphragm moves toward the backplate,

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fluid is squeezed out of the near region, into the regions further from the diaphragm, and viscous dissipation occurs. If the viscous penetration depth δ is small compared with h, the fluid flow will involve energy losses primarily near the diaphragm and near the backplates; on the other hand, if $\delta \gg h$, one has a slowly modulated form of Poiseuille flow, in which the losses occur throughout the volume of the system.

We will model this problem for three geometries. First, we will consider two-dimensional flow through a symmetrical channel whose walls are uniformly oscillating at an angular frequency ω; secondly, the separation between two circular plates will be varied; and thirdly, the radius of a pipe will be varied. In each case it will be assumed that there is a fluid reservoir at pressure p_0 so that the fluid has room to move, and thus behaves as if it is incompressible. (This requires that the characteristic wavelength $\lambda = 2\pi c/\omega$ of sound be large compared to the characteristic channel length. Here c is the sound velocity.) In addition, we will consider only small-amplitude oscillations, so that the equations may be linearized.

Some work has been done which bears on the present paper. Secomb (1978) has studied small-amplitude oscillations in the channel and pipe geometries, with an interest in arterial blood flow. The pipe geometry has also been studied, for monotonic compression and expansion, including finite amplitude effects, by Uchida & Aoki (1977); they also had an interest in blood flow. The disk geometry, with one disk fixed, has been studied also. First, slow compression (or expansion) is treated in Landau & Liftshitz (1959), being attributed to Reynolds. In addition, Terrill (1969) has treated the case where one disk oscillates, including some finite-amplitude effects; his interest was in problems associated with lubrication. None of these papers consider either power loss or acoustic impedance, which is the primary concern of the present paper.

2. Flow solutions

We will look for solutions to the linearized mass and momentum conservation equations, considering the fluid to be incompressible. Thus we write the fluid velocity as

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2, \tag{1}$$

where

$$\mathbf{v}_1 = \nabla\phi, \quad \nabla^2\phi = 0 \tag{2}$$

describes the longitudinal part of the solution, and

$$\eta\nabla^2\mathbf{v}_2 = \rho\frac{\partial \mathbf{v}_2}{\partial t} = -i\omega\rho\mathbf{v}_2, \quad \nabla.\mathbf{v}_2 = 0. \tag{3}$$

describes the transverse part of the solution. (In (3) we have taken $\mathbf{v}_2$ to vary as $e^{-i\omega t}$.)

Rather than derive in detail the solutions for the various geometries, we will merely state what the solutions are, considering the geometries in succession.

(a) *Symmetrical channel*

Consider a channel of height $2a$ and length $2l$. We will assume symmetrical flow about its centre, so that under compression of the plates, fluid moves to the right or left according to whether it is to the right or left of the midpoint of the plates. We will also impose the boundary condition that, at the plate boundaries, the fluid displacement $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi}(x, y = \pm a) = \pm \hat{\mathbf{j}} \xi e^{-i\omega t}, \tag{4}$$

where $\xi \ll a$, and the $\pm$ denotes whether we are at the upper or lower plate. Since $\mathbf{v} = \partial \boldsymbol{\xi}/\partial t$, this becomes the condition that

$$\mathbf{v}(x, y = \pm a) = \pm \hat{\mathbf{j}}(-i\omega\xi)\, e^{-i\omega t}. \tag{5}$$

The solution to the differential equations, subject to the boundary condition given by (5), is (with $v_0 = -i\omega\xi e^{-i\omega t}$)

$$v_x = Ax(1 - \cos\alpha y/\cos\alpha a), \tag{6}$$

$$v_y = -A(y - \alpha^{-1}\sin\alpha y/\cos\alpha a), \tag{7}$$

where

$$A = v_0(\alpha^{-1}\tan\alpha a - a)^{-1}. \tag{8}$$

Since α is a complex number, these equations hide some of the complexity in this problem, but it is helpful that they have so compact a form. In the static limit ($\omega \to 0$ but finite v_0), we have

$$v_x \to v_0[3x(y^2 - a^2)/2a^3], \tag{9}$$

$$v_y \to v_0[y(3a^2 - y^2)/2a^3]. \tag{10}$$

Note that (9) and (10) agree with § 4 of Secomb (1978), when the different notations are accounted for.

(*b*) *Circular disks*

In this case the vertical disk separation is $2a$ and the disk radius is l. The solution to the differential equations, subject to the boundary conditions that $v_y(r, y = \pm a) = \pm v_0$ and $v_r(r, y = \pm a) = 0$, is

$$v_r = Br(1 - \cos\alpha y/\cos\alpha a), \tag{11}$$

$$v_y = -2B(y - \alpha^{-1}\sin\alpha y/\cos\alpha a), \tag{12}$$

where

$$B = \tfrac{1}{2}v_0(\alpha^{-1}\tan\alpha a - a)^{-1}. \tag{13}$$

In the static limit we have

$$v_r \to v_0[3r(y^2 - a^2)/4a^3], \tag{14}$$

$$v_y \to v_0[y(3a^2 - y^2)/2a^3]. \tag{15}$$

(*c*) *Pipe of oscillating radius*

In this case let the radius of the tube be given by a, and let the tube have length $2l$ along the z direction. The solution to the differential equations, subject to the boundary conditions $v_r(z, r = a) = v_0$ and $v_z(z, r = a) = 0$ is given in terms of Bessel functions as

$$v_z = Cz[1 - J_0(\alpha r)/J_0(\alpha a)], \tag{16}$$

$$v_r = -\tfrac{1}{2}C[r - 2\alpha^{-1}J_1(\alpha r)/J_0(\alpha a)], \tag{17}$$

where
$$C = v_0[\alpha^{-1}J_1(\alpha a)/J_0(\alpha a) - \tfrac{1}{2}a]. \tag{18}$$
In the static limit we have
$$v_z \to v_0[4z(r^2 - a^2)/a^3], \tag{19}$$
$$v_r \to v_0[r(2a^2 - r^2)/a^3]. \tag{20}$$
Note that (15) and (16) agree with the appendix of Secomb (1978), when the different notations are accounted for.

3. Stress tensor

We will continue to develop the solutions to each of the three problems by treating their various aspects in parallel. In this section we will compute the appropriate stress tensors and pressures. We will employ the linearized Navier–Stokes equation,
$$-i\omega\rho\mathbf{v} = -\nabla p + \eta\nabla^2\mathbf{v} = \nabla.\boldsymbol{\sigma}, \tag{21}$$
where $\boldsymbol{\sigma}$ is the stress tensor, given (for $\nabla.\mathbf{v} \approx 0$) by
$$\sigma_{ij} = -p\delta_{ij} + \eta(\partial_i v_j + \partial_j v_i). \tag{22}$$
Knowledge of $\mathbf{v}$ and (21) will permit us to deduce p; knowledge of p, $\mathbf{v}$ and (22) gives $\boldsymbol{\sigma}$.

We will determine p by solving
$$\nabla p = i\omega\rho\mathbf{v} + \eta\nabla^2\mathbf{v}. \tag{23}$$

(a) *Symmetrical channel*

Here we must solve
$$\partial_x p = i\omega\rho v_x + \eta(\partial_x^2 + \partial_y^2)\, v_x, \tag{24}$$
$$\partial_y p = i\omega\rho v_y + \eta(\partial_x^2 + \partial_y^2)\, v_y. \tag{25}$$
Placing (6) into (24) we find that
$$\begin{aligned}\partial_x p &= i\omega\rho Ax(1 - \cos\alpha y/\cos\alpha a) + \eta Ax\alpha^2\cos\alpha y/\cos\alpha a\\ &= i\omega\rho Ax,\\ p &= \tfrac{1}{2}i\omega\rho Ax^2 + f(y,t),\end{aligned} \tag{26}$$
where $f(y,t)$ must be determined. Placing (7) into (25) yields
$$\begin{aligned}\partial_y p &= -i\omega\rho A(y - \alpha^{-1}\sin\alpha y/\cos\alpha a) - \eta A\alpha\sin\alpha y/\cos\alpha a\\ &= -i\omega\rho Ay,\\ p &= -\tfrac{1}{2}i\omega\rho Ay^2 + g(x,t).\end{aligned} \tag{27}$$
Clearly, the solution to (26) and (27), subject to $p = p_0$ for $(x,y) = (l,0)$, is given by
$$p = p_0 + \tfrac{1}{2}i\omega\rho A(x^2 - l^2 - y^2). \tag{28}$$

For completeness, we must also compute σ_{yy}. It is given by
$$\begin{aligned}\sigma_{yy} &= -p + 2\eta\,\partial_y v_y\\ &= -p_0 - \tfrac{1}{2}i\omega\rho A(x^2 - l^2 - y^2) - 2\eta A(1 - \cos\alpha y/\cos\alpha a).\end{aligned} \tag{29}$$

The rate of work dP being done on the system, per unit area dA, is given by

$$dP/dA = \sigma_{yy} v_y. \tag{30}$$

This must be integrated over both top and bottom surfaces. Since it is the time-averaged quantity in which we shall be most interested, we must take only the real parts in computing σ_{yy} and v_y. To maintain the parallelism, we shall reserve this calculation for the next section.

(b) *Circular disks*

Here we will merely state our final results. The pressure, obtained from solving (23), and subject to $p = p_0$ for $(r, y) = (l, 0)$, is

$$p = p_0 + \tfrac{1}{2} i\omega\rho B(r^2 - l^2 - 2y^2). \tag{31}$$

The rate of work being done on the system, per unit area, is given by

$$dP/dA = \sigma_{yy} v_y, \tag{32}$$

and σ_{yy} is given by

$$\begin{aligned}\sigma_{yy} &= -p + 2\eta\, \partial_y v_y \\ &= -p_0 - \tfrac{1}{2} i\omega\rho B(r^2 - l^2 - 2y^2) - 4\eta B(1 - \cos\alpha y/\cos\alpha a).\end{aligned} \tag{33}$$

Again, we defer the calculation of the power input until the next section.

(c) *Pipe*

Again we merely state our final results. The pressure p, obtained from solving (23), and subject to $p = p_0$ for $(z, r) = (l, 0)$, is

$$p = p_0 + \tfrac{1}{2} i\omega\rho C(z^2 - l^2 - \tfrac{1}{2} r^2). \tag{34}$$

Note that (34) agrees with the result of Uchida & Aoki (1977), in the limit of small radial velocity. The rate of work being done on the system, per unit area, is given by

$$dP/dA = \sigma_{rr} v_r, \tag{35}$$

$$\sigma_{rr} = -p + 2\eta\, \partial_r v_r. \tag{36}$$

From (34) and (17) we have

$$\sigma_{rr} = -p_0 - \tfrac{1}{2} i\omega\rho C(z^2 - l^2 - r^2) - \eta C[1 - 2J_1'(\alpha r)/J_0(\alpha a)], \tag{37}$$

where $J_1'(\alpha r)$ is the derivative of $J_1(\alpha r)$ with respect to its argument. From

$$\partial J_1/\partial x + J_1/x = J_0,$$

where x is the argument of J_1 and J_0, we may rewrite (37) as

$$\sigma_{rr} = -p_0 - \tfrac{1}{2} i\omega\rho C(z^2 - l^2 - \tfrac{1}{2} r^2) - \eta C\{1 - 2[J_0(\alpha r) - \alpha^{-1} r^{-1} J_1(\alpha r)]/J_0(\alpha a)\}. \tag{38}$$

Again we defer the calculation of the power input until the next section.

4. Power input

In what follows, we will compute the power input, a quantity which is second order in the velocity. Because our first-order solution *exactly* matches the boundary condition of a moving wall, the boundary condition on the second-order solution $\mathbf{v}^{(2)}$ is that $\mathbf{v}^{(2)} = 0$ on the wall. (This is true for all the higher-order solutions.) Hence it cannot contribute to second order, since the product of the zeroth-order stress tensor and $\mathbf{v}^{(2)}$ then gives zero on the wall, and the product of the second-order stress tensor and the zeroth-order velocity $\mathbf{v}^{(0)}$ is also zero on the wall, since $\mathbf{v}^{(0)} = 0$ everywhere. Only the product of the first-order stress tensor and $\mathbf{v}^{(1)}$ needs to be considered in computing the power input to second order. We also note that our linearized analysis will be valid as long as the amplitude ξ is much less than the viscous penetration depth δ (i.e., $\xi \ll \delta$). For $\xi \gtrsim \delta$, gradients of the first-order (i.e., linearized) solutions, which appear as source terms in the equation for the second-order solutions, can cause the second-order solutions to be comparable in magnitude to the first-order solutions.

(a) *Symmetrical channel*

As a start, let us first consider the power needed to compress the channel as $\omega \to 0$. In that case, $\alpha \to 0$, so that $A \to 3v_0/\alpha^2 a^3$, and thus, for $y = a$,

$$\sigma_{yy} \to -p_0 - \tfrac{3}{2}(\eta v_0/a^3)\,(x^2 - l^2 - a^2). \tag{39}$$

Since the compression is uniform, we can find the power provided by each surface simply by multiplying the force on each surface by the area of each surface. For this geometry we will work with the force of compression f per unit length dL along the z axis. Thus, neglecting the a^2 term in (39),

$$f = \int_{-l}^{l} \sigma_{yy}\,dx = -2l[p_0 - \eta v_0 l^2/a^3], \tag{40}$$

$$dP/dL = 2fv_0 = -4p_0 v_0 l + 4\eta v_0^2 l^3/a^3. \tag{41}$$

Since v_0 is negative in compression, this is indeed a positive quantity. Besides the $p_0 v_0$ term, which would not appear if the channel were immersed in fluid on all sides (due to compensating stresses on the outer walls of the channel), there is an additional viscous term. (Properly, the a^2 term in (39) cannot be considered to be given accurately, since the fluid flow pattern is known only if $l \gg a$.)

Let us now compute the finite frequency power input. For $y = a$, (29) and (8) give

$$\sigma_{yy} \approx -p_0 - \tfrac{1}{2}i\omega\rho v_0(\alpha^{-1}\tan\alpha a - a)^{-1}(x^2 - l^2). \tag{42}$$

To evaluate the time-average of the power per unit area dP/dA, we note, for an oscillation of two complex quantities A and B, each of period T, that

$$T^{-1}\int_0^T \mathrm{Re}\,(A) \times \mathrm{Re}\,(B)\,dt = \tfrac{1}{2}\,\mathrm{Re}\,(AB^*). \tag{43}$$

Hence (30), becomes where $\bar{A}$ denotes the time-average of A,

$$\begin{aligned}\overline{dP/dA} &= \overline{\sigma_{yy} v_y} = \tfrac{1}{2}\,\mathrm{Re}\,(\sigma_{yy} v_y^*) \\ &= \tfrac{1}{4}\omega\rho|v_0|^2(x^2 - l^2)\,\mathrm{Im}\,[(\alpha^{-1}\tan\alpha a - a)^{-1}],\end{aligned} \tag{44}$$

where $\alpha = (i\omega\rho/\eta)^{\frac{1}{2}} = (1+i)/\delta$, with $\delta \equiv (2\eta/\omega\rho)^{\frac{1}{2}}$ being the viscous penetration depth. Integrating from $-l$ to $+l$, and including top and bottom, we have

$$\overline{dP/dL} = -\tfrac{4}{3}\omega\rho|v_0|^2 l^3 \operatorname{Im}[(\alpha^{-1}\tan\alpha a - a)^{-1}], \tag{45}$$

which agrees with the second term of (41) as $\omega \to 0$. An explicit expression for the bracketed term in (45) can be obtained after some tedious algebra, but it provides little illumination. In the high-frequency limit (by which we mean that $|\alpha|a \gg 1$, or $a \gg \delta$, so that potential flow dominates), it takes the form

$$(\alpha^{-1}\tan\alpha a - a)^{-1} \to -a^{-1}[1+(1+i)\,\delta/2a],$$

so that

$$\overline{dP/dL} \to \tfrac{2}{3}\omega\rho|v_0|^2 l^3\delta/a^2 = \tfrac{1}{3}(2\eta\rho\omega)^{\frac{1}{2}}\,|v_0|^2 l^3/a^2. \tag{46}$$

This differs from the $a \ll \delta$ limit (of (41)) by a factor of $(a/6\delta)$. Hence the time-averaged power dissipation can be expected to slowly rise as the frequency increases, provided that $|v_0|$ is independent of frequency. On the other hand, if the amplitude ξ is independent of frequency, then $|v_0|$ varies as ω, and the time-averaged power dissipation rises much more rapidly, starting as ω^2 at low ω, and going as $\omega^{\frac{5}{2}}$ at high ω.

(*b*) *Circular disks*

For $y = a$ and $r, l \gg a$, (33) and (13) give

$$\sigma_{yy} \approx -p_0 - \tfrac{1}{4}i\omega\rho v_0(\alpha^{-1}\tan\alpha a - a)^{-1}(r^2 - l^2). \tag{47}$$

Integrating this over the area of a disk gives

$$F_y = \int_0^l \sigma_{yy}(2\pi r\,dr) = -\pi p_0 l^2 + i\frac{\pi}{8}\,\omega\rho v_0 l^4(\alpha^{-1}\tan\alpha a - a)^{-1}. \tag{48}$$

In the $\omega \to 0$ limit this gives

$$F_y \to -\pi p_0 l^2 + \tfrac{3}{8}\pi\eta v_0 l^4/a^3, \tag{49}$$

$$P = 2F_y v_0 = -2\pi p_0 v_0 l^2 + \tfrac{3}{4}\pi\eta v_0^2 l^4/a^3. \tag{50}$$

In the high-frequency limit (48) gives (neglecting the p_0 term)

$$F_y \to -\tfrac{1}{8}i\pi\omega\rho v_0 l^4(a^{-1} + i\delta/2a^2), \tag{51}$$

$$\begin{aligned}\overline{P} &= 2\,\overline{\operatorname{Re}(F_y)\operatorname{Re}(v_0)} = \operatorname{Re}(F_y v_0^*)\\ &= \tfrac{1}{16}\pi(2\eta\rho\omega)^{\frac{1}{2}}\,|v_0|^2 l^4/a^2.\end{aligned} \tag{52}$$

More generally, we have, for finite ω, that

$$\overline{P} = \operatorname{Re}(F_y v_0^*) = -\tfrac{1}{8}\pi\omega\rho|v_0|^2 l^4 \operatorname{Im}[(\alpha^{-1}\tan\alpha a - a)^{-1}]. \tag{53}$$

(*c*) *Pipe*

For $r = a$ and $z, l \gg a$, (18) and (38) give

$$\sigma_{rr} \approx -p_0 - \tfrac{1}{2}i\omega\rho v_0[\alpha^{-1}J_1(\alpha a)/J_0(\alpha a) - \tfrac{1}{2}a]^{-1}(z^2 - l^2). \tag{54}$$

Integrating this over the area of the pipe gives

$$F_r = \int_{-l}^{l} \sigma_{rr}\, 2\pi a\, dz$$
$$= -4\pi p_0 al + \tfrac{4}{3}\pi i\omega\rho v_0 l^3 a[\alpha^{-1}J_1(\alpha a)/J_0(\alpha a) - \tfrac{1}{2}a]^{-1}. \quad (55)$$

In the $\omega \to 0$ limit this gives

$$F_r \to -4\pi p_0 al + \frac{64\pi\eta v_0 l^3}{3a^2}. \quad (56)$$

In the high-frequency limit, where $a \gg \delta$, (55) requires that we employ, for $|z| \to \infty$ and $z = |z|\, e^{i\phi}$ with $-\tfrac{1}{2}\pi < \phi < \tfrac{1}{2}\pi$,

$$J_n(z) \to (2/\pi z)^{\frac{1}{2}} \cos[z - \tfrac{1}{2}\pi(\eta + \tfrac{1}{2})]. \quad (57)$$

Here we have $z = \alpha a = (\sqrt{2}\, a/\delta)\, e^{\frac{1}{4}i\pi} = (a/\delta)\,(1+i)$, so

$$J_n(\alpha a) \to (\sqrt{2}\, \delta\, e^{-\frac{1}{4}i\pi}/\pi a)^{\frac{1}{2}} \cos[2/\delta - \tfrac{1}{2}\pi(n+\tfrac{1}{2}) + ia/\delta]$$
$$\to (\sqrt{2}\, \delta/4\pi a)^{\frac{1}{2}} \exp\{-i[a/\delta - \tfrac{1}{2}\pi n] + a/\delta\}. \quad (58)$$

Thus, for $a \gg \delta$,

$$[\alpha^{-1}J_1(\alpha a)/J_0(\alpha a) - \tfrac{1}{2}a]^{-1} \approx (2/a)\,[\sqrt{2}\,(\delta/a)\, e^{\frac{1}{4}i\pi} - 1]^{-1}$$
$$\approx -(2/a)\,(1 + i\delta/a). \quad (59)$$

Hence, neglecting the p_0 term, in the high-frequency limit (55) gives

$$F_r \to -\tfrac{8}{3} i\pi\omega\rho v_0 l^3 (1 + i\delta/a), \quad (60)$$

$$\overline{P} = \overline{\mathrm{Re}\,(F_r)\,\mathrm{Re}\,(v_0)} = \tfrac{1}{2}\,\mathrm{Re}\,(F_r v_3^*)$$
$$= \tfrac{4}{3}\pi(2\eta\rho\omega)^{\frac{1}{2}}\, |v_0|^2 l^3/a. \quad (61)$$

More generally we have, for finite ω, that

$$U = -\tfrac{8}{3}\pi\omega\rho |v_0|^2 l^3 a\, \mathrm{Im}\,\{[\alpha^{-1}J_1(\alpha a)/J_0(\alpha a) - \tfrac{1}{2}a]^{-1}\}. \quad (62)$$

5. Acoustic impedance

Because this work was originally motivated by a problem in acoustics, we will take our calculations one step further, and compute the average acoustic impedance

$$z \equiv p/v. \quad (63)$$

Here p is the average force per unit area, excluding the background pressure, and v is the velocity of the surface. A large value of z indicates a surface that is hard to move. Note that in what follows, we must employ $z = -p/v$, since for our geometries a positive v corresponds to a decrease in p, and we must have $\mathrm{Re}\,(z) > 0$ for energy absorption.

We first consider the low-frequency values. For the symmetrical channel, (40) gives

$$z = \eta l^2/a^3. \quad (64)$$

For the circular disks, (49) gives

$$z = \tfrac{3}{8}\eta l^2/a^3. \quad (65)$$

For the pipe, (56) gives

$$z = \tfrac{16}{3}\eta l^2/a^3. \quad (66)$$

	ρc	η	$\eta l^2/a^3$
Air (STP)	$4 \cdot 0 \times 10^1$	$1 \cdot 8 \times 10^{-4}$	$1 \cdot 8 \times 10^2$
Water (STP)	$1 \cdot 5 \times 10^5$	$1 \cdot 0 \times 10^{-2}$	$1 \cdot 0 \times 10^4$
^{3}He (20 mK, $P = 0$)†	$1 \cdot 5 \times 10^3$	$4 \cdot 6 \times 10^{-3}$	$4 \cdot 6 \times 10^3$

Here $l = 1$ cm, $a = 10^{-2}$ cm.
† Wheatley 1975.

TABLE 1

	η/ρ	$\delta = (2\eta/\omega\rho)^{\frac{1}{2}}$	$\omega\rho l^2/a$	$\omega\rho l^2\delta/a^2$
Air (STP)	$1 \cdot 5 \times 10^{-1}$	$2 \cdot 2 \times 10^{-3}$	$7 \cdot 5 \times 10^3$	$1 \cdot 7 \times 10^3$
Water (STP)	$1 \cdot 0 \times 10^{-2}$	$5 \cdot 6 \times 10^{-4}$	$6 \cdot 3 \times 10^6$	$3 \cdot 5 \times 10^5$
^{3}He (20 mK, $P = 0$)†	$5 \cdot 6 \times 10^{-2}$	$1 \cdot 3 \times 10^{-3}$	$5 \cdot 2 \times 10^5$	$6 \cdot 7 \times 10^4$

Here $\omega = 2\pi \times 10^4$ s^{-1}, $l = 1$ cm, $a = 10$ cm.
† Wheatley 1975.

TABLE 2

For air at room temperature and atmosphere pressure (STP) we have $\eta = 1 \cdot 8 \times 10^{-4}$ cgs, so that if $l = 1$ cm and $a = 10^{-2}$ cm, $\eta l^2/a^3 = 180$ cgs. This is to be compared to ρc, the usual bulk acoustic impedance, which for air at STP is about 40 cgs. In other words, if the fluid being compressed is confined to a narrow region, it has a large impedance, or resistance, to compression, simply due to viscous effects. This is well known in the study of lubrication. The values for water at STP, and for liquid ^{3}He at 20×10^{-3} K and zero pressure are given in table 1, for the same geometry.

We now consider the high-frequency limit, where $a \gg \delta \equiv (2\eta/\omega\rho)^{\frac{1}{2}}$. As a standard frequency we will take 10^4 Hz, a not uncommon acoustic frequency, although it is considerably higher than what one could confront in a mechanical system (such as a pump). The acoustic impedance for the symmetrical channel becomes, from (42) averaged from $-l$ to $+l$,

$$z = -i\tfrac{1}{3}\omega\rho(l^2/a)(1+i\delta/2a). \tag{67}$$

The imaginary part is an inertial effect due to potential flow induced in the fluid. The real part corresponds to actual losses in the system. For a circular disk, from (51) we have

$$z = -i\tfrac{1}{8}\omega\rho(l^2/a)(1+i\delta/2a). \tag{68}$$

For a pipe, from (60) we have

$$z = -i\tfrac{2}{3}\omega\rho(l^2/a)(1+i\delta/a). \tag{69}$$

Representative values of $\omega\rho l^2/a$ and $\omega\rho l^2\delta/a^2$ are given in the last two columns of table 2.

Comparing ρc and $\omega\rho l^2\delta/a^2$ for liquid ^{3}He at 20 mK, $P = 0$, and 10^4 Hz, we see that the latter quantity, due to viscous loss, is forty times as large as the former quantity, due to acoustic transmission. Clearly, under such circumstances it would be necessary to include the viscous contribution to the total impedance. Probably it would be preferable, in the context of condenser microphone design, to eliminate such effects as much as possible. Nevertheless, it may often be of value to have (equations (67)–(69)) simply in order to determine when such effects become important.

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